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ON THE APPROXIMATE SOLUTION OF INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE

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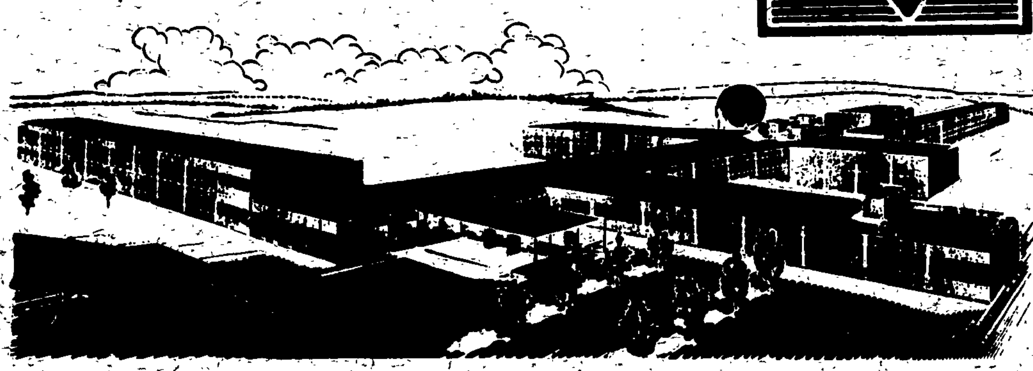

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(6) ON THE APPROXIMATE SOLUTION OF INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE

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FOREWORD

This final technical report covers research initiated by the Aeronautical Research Laboratories, Office of Aerospace Research, United States Air Force on Contract AF 33(616)-7062, Project 7071, "Mathematical Techniques of Aeromechanics," Task 7071-02, "Methods of Mathematical Physics." The work was under the cognizance of the Applied Mathematics Laboratory with Mr. H. E. Fettis acting as project director.

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ABSTRACT

Convolution integral equations have a simple representation in the form of a Laplace transform, and so solution follows by transform inversion. Usually, the solution is expressed as an infinite integral which can only be evaluated by numerical integration. Often, the transform involves the ratio of functions of hypergeometric type. ~~In previous papers (see "The Padé Table and the τ-Method," Jour. Math. Phys., 37, 110-127 (1958) and "On Economic Representations of Transcendental Functions," Jour. Math. Phys., 38, 279-294 (1960)), the author studied rational approximations for these transcendents. Here we study~~ approximations to an inverse Laplace transform* when the transform, which involves the exponential integral, is approximated by rational functions. The approximate solution is a sum of exponential functions, and numerics are presented to show the efficiency of our technique.

*are studied

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INTRODUCTION

Numerous problems in applied mathematics require the solution of integral equations of the convolution type. It is well known that equations of this kind have a rather simple representation in the form of a Laplace transform so that the solution depends upon finding an inverse Laplace transform. Now it may be that the Laplace transform involves functions for which rational approximations are available, and it is of interest to study the nature of approximations to inverse Laplace transforms when the transforms are approximated by rational functions.

In previous studies [1,2], we have been concerned with rational approximations to higher transcendental functions. The approximations are, of course, useful to evaluate the transcendental functions as in the sense of table making. However, a more cogent reason for investigating such approximations is that they should be useful, for example, to invert Laplace transforms, and in general to further simplify the evaluation of mathematical solutions to applied problems already expressed in closed form.

In two recent studies [3,4], we have investigated problems in applied mechanics and aerodynamics whose solution requires inversion of a Laplace transform which involves ratios of functions of hypergeometric type. In the cases studied, the hypergeometric functions were the modified Bessel functions of the second kind $K_\nu(z)$. With the aid of results developed in [2], we derived economic approximations for the inverse functions. In the present paper, we present a similar study where the transcendent involved is the exponential integral.

I. SOLUTION OF A CONVOLUTION INTEGRAL EQUATION

For our particular study, we take an example from a paper by Friedlander [5] which requires the solution of

$$\int_0^z k(z-t)g(t)dt = g(z)+f(z) \quad ,$$

$$k(z) = 2f(z) = -2(z+2)^{-2} \quad . \tag{1.1}$$

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Using the theory of Laplace transforms, we have

$$G(p) = \frac{F(p)}{K(p)-1} \ , \ K(p) = 2F(p) \quad ,$$

$$F(p) = -\frac{1}{2} - pEi(-2p) \ , \ -Ei(-2p) = \int_{2p}^{\infty} t^{-1}e^{-t}dt \quad ,$$

$$G(p) = \frac{\frac{1}{2}\left\{1+2pe^{2p}Ei(-2p)\right\}}{2+2pe^{2p}Ei(-2p)} \quad . \tag{1.2}$$

Here $F(p)$, $G(p)$ and $K(p)$ are Laplace transforms of $f(z)$, $g(z)$ and $k(z)$, respectively. Also $-Ei(-z)$ is the usual notation for the exponential integral. By the inversion formula for Laplace transforms,

$$g(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pz}G(p)dp \quad , \tag{1.3}$$

where $c > 0$ and c lies to the right of all singularities of $G(p)$.

In the remainder of this section, we develop some results which lead to an integral representation of $g(z)$. The results obtained are also useful in connection with our process for approximating $g(z)$. We first show that $G(p)$ has no poles in the complex plane cut along the negative real axis, and that $G(p)$ is analytic everywhere except on this axis for at the origin it has a branch point. To prove this, we first note that

$$-Ei(-z)+(\gamma+\ln z) = \int_{0}^{z} t^{-1}(1-e^{-t})dt = -\sum_{k=1}^{\infty}\frac{(-)^{k}z^{k}}{k!k} \quad , \tag{1.4}$$

$$Ei(x) = \int_{-\infty}^{x} t^{-1}e^{t}dt \quad , \quad x > 0 \quad , \tag{1.5}$$

$$\mathrm{Ei}(z) = \mathrm{Ei}\left(-ze^{\pm i\pi}\right) \mp i\pi = \frac{1}{2}\left[\mathrm{Ei}\left(-ze^{i\pi}\right) + \mathrm{Ei}\left(-ze^{-i\pi}\right)\right] , \tag{1.6}$$

$$-ze^{z}\mathrm{Ei}(-z) \sim {}_2F_0(1,1;-z^{-1}) = \sum_{k=0}^{\infty} (-)^k k!\, z^{-k} , \quad |z| \to \infty , \quad |\arg z| < 3\pi/2 . \tag{1.7}$$

In (1.4), γ is Euler's constant, and in (1.5) the integral is a Cauchy principal value. Also in (1.7) and in the sequel, standard generalized hypergeometric notation is used. Thus

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots a_p \\ b_1, b_2, \ldots b_q \end{matrix} \middle| z\right) = {}_pF_q(a_1, a_2, \ldots a_p; b_1, b_2, \ldots b_q; z)$$

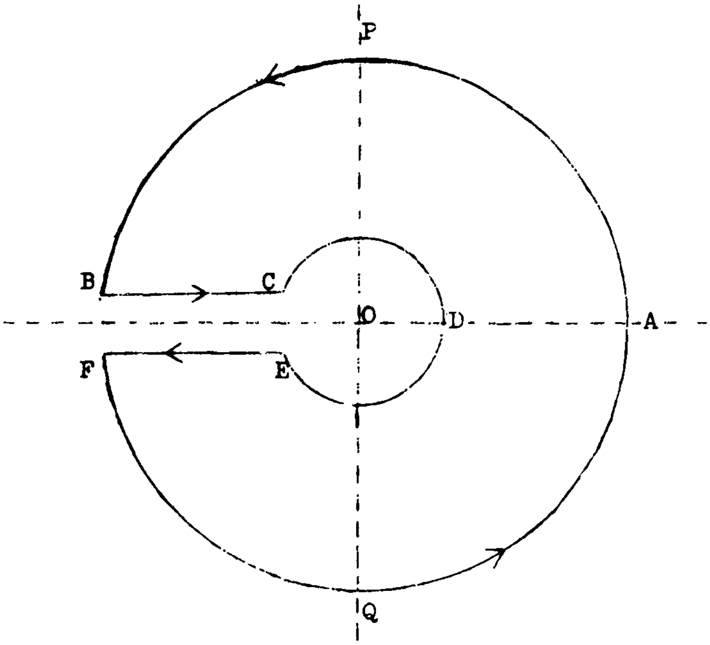
$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_k z^k}{\prod_{i=1}^{q} (b_i)_k k!} , \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} . \tag{1.8}$$

For further discussion of hypergeometric functions, see [6].

Now let

$$h(z) = 1 + \frac{1}{2}(2z)e^{2z}\mathrm{Ei}(-2z) \tag{1.9}$$

and consider the contour described by the figure below.



Here let Γ_1 be the small circle with radius r, and Γ_2 the large circle with radius R. The complex plane is cut along the negative real axis so that along BC, $\arg z$ is π, while along EF, $\arg z$ is $-\pi$. Then the number of zeros of $h(z)$ which lie within the contour is $(2\pi)^{-1}$ times the change in phase of $h(z)$ as z traverses the contour. Now the change in phase is

$$\Big[\arg h(z)\Big]_{\Gamma_2} - \Big[\arg h(z)\Big]_{\Gamma_1} + \Big[\arg h(z)\Big]_{Re^{i\pi}}^{re^{i\pi}} + \Big[\arg h(z)\Big]_{re^{-i\pi}}^{Re^{-i\pi}} \ . \qquad (1.10)$$

As $R\to\infty$ and $r\to 0$, the first two terms vanish since $\lim_{z\to 0} h(z) = \lim_{z\to\infty} h(z) = 1$. Now a representation for $h\left(xe^{\pm i\pi}\right)$ follows from (1.6), and so the last two terms of (1.9) become

$$2\pi \lim_{\substack{r\to 0\\ R\to\infty}} \left[\text{arc tan} \frac{xe^{-2x}}{1-xe^{-2x}\text{Ei}(2x)}\right]_r^R = 0 \ . \qquad (1.11)$$

Thus $g(z)$ has no zeros in the complex plane cut along the negative real axis.

It readily follows that the path of integration for (1.3) may be deformed into the path QAP with $R \to \infty$ and that the value of this integral taken over the complete circuit APBCDEFQA is zero. The integral over the arcs PB and FQ tend to zero as $R \to \infty$, and so we have the loop integral representation

$$g(z) = \frac{1}{2\pi i} \int_{-\infty}^{(o+)} e^{pz} G(p) dp \quad , \tag{1.12}$$

where $\int_{-\infty}^{(o+)}$ means that the path of integration is FEDCB with $r \to 0$ and $R \to \infty$. We readily find the integral representation

$$g(z) = \frac{1}{4} \int_0^\infty \frac{x e^{-x(z+2)} dx}{\left\{1 - x e^{-2x} Ei(2x)\right\}^2 + \pi^2 x^2 e^{-4x}} \quad , \; Re(z) > -2 \quad , \tag{1.13}$$

where Re stands for the real part.

At this point, it is helpful to give a synopsis of the material in the remaining sections. In Section II, we develop the usual power series solution of $g(z)$ for small z, and an asymptotic expansion of $g(z)$ for large z. It is the method of constructing the power series solution that motivates our idea of using rational approximations to approximate $G(p)$ which in turn leads to an approximation of $g(z)$. The method of approximation including a complete error analysis is the subject of Section III. Numerical examples are given in Section IV.

II. THE CONVENTIONAL SOLUTIONS FOR SMALL AND LARGE VARIABLE

To get an expansion in ascending powers of z, we combine (1.7) and (1.2) and formally develop $G(p)$ as a series in reciprocal powers of p. Termwise inversion then produces the desired result. Numerous examples illustrating this technique have been given by Goldstein [7]. See also Carslaw and Jaeger [8]. We find that

$$g(z) = \frac{1}{4} \sum_{k=0}^{\infty} \frac{(-)^k a_k z^k}{2^k k!} \quad , \quad a_0 = 1 \quad ,$$

$$a_k = (k+1)! + \sum_{r=0}^{k-1} (k-r)! a_r \quad . \tag{2.1}$$

Further numerical coefficients are given in the following table.

k	a_k	k	a_k
1	3	6	8431
2	11	7	62391
3	47	8	5 24495
4	231	9	49 60775
5	1303	10	522 23775

(2.2)

The series (2.1) converges quite slowly. Indeed, it is clear from (1.13) and a theorem in complex variable theory that (2.1) converges only if $|z| < 2$. Thus, if $z = \frac{1}{2}$ and we use 11 terms of (2.1), the result is only correct to about five decimals. Again, at $z = 1$, we only get accuracy to about 2 decimals.

To get an asymptotic expansion of $g(z)$, we use (1.4) to develop an ascending series for $G(p)$, and then employ (1.12). For other examples of this approach see Carslaw and Jaeger [8], and Ritchie and Sakakura [9]. We have

$$4G(p) = 1 + pe^{2p}(\ln p+\epsilon) - p^2e^{2p}\left\{2 + e^{2p}(\ln p+\epsilon)^2\right\}$$
$$+ p^3e^{2p}\left\{1 + 4e^{2p}(\ln p+\epsilon) + e^{4p}(\ln p+\epsilon)^3\right\}+\ldots,$$
$$\epsilon = \gamma + \ln 2 \quad . \tag{2.3}$$

Now

$$b_\nu = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{\lambda p}p^{\nu-1}dp = \frac{\Gamma(\nu)\sin \nu\pi}{\pi\lambda^\nu} \quad , \quad R(\nu) \geq 0 \quad ,$$
$$b_0 = 1 \quad , \quad b_n = 0 \quad , \quad n \text{ a positive integer} \quad , \tag{2.4}$$

and

$$c_{mn} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{\lambda p}p^{n-1}(\ln p)^m dp = \left(\frac{\partial^m b_\nu}{\partial\nu^m}\right)_{\nu=n} \quad ,$$
$$c_{1n} = \frac{(-)^n(n-1)!}{\lambda^n} \quad , \quad c_{2n} = \frac{2(-)^n(n-1)!}{\lambda^n}\left[\psi(n)-\ln \lambda\right] \quad ,$$
$$c_{3n} = \frac{(-)^n(n-1)!}{\lambda^n}\left[3\left\{\psi(n)-\ln \lambda\right\}^2 + 3\psi'(n) - \pi^2\right] \quad , \text{ etc. } , \tag{2.5}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function $\Gamma(z)$ and $\psi'(z)$ is the derivative of $\psi(z)$. The union of (2.3) and (1.12) gives

$$g(z) \sim \frac{1}{4(z+2)^2} + \frac{\frac{3}{2} - \ln\left(\frac{z+4}{2}\right)}{(z+4)^3} + \frac{6}{(z+4)^4}$$
$$+ \frac{9}{2(z+6)^4}\left[-\frac{\pi^2}{6} +2 - \frac{11}{3}\ln\left(\frac{z+6}{2}\right)+\left\{\ln\left(\frac{z+6}{2}\right)\right\}^2\right]$$
$$+ \ldots, \quad 0 < z \to\infty \quad . \tag{2.6}$$

Note that (2.6) is not very effective unless z is quite large, say $z > 6$. Also, the formula is deficient since a useful estimate of the error is not at hand, although this drawback is not too serious.

The representations (2.1) and (2.6) are therefore only useful for z very small or z very large. There is need for analytical representations to cover the intermediate range. We shall show how rational approximations to $-Ei(-z)$ can be effectively and economically used to approximate $g(z)$ for z ranging from the origin into the region where use of the asymptotic expansion is meaningful.

III. SOLUTION BY RATIONAL APPROXIMATIONS

It is known that the exponential integral may be approximated by a sequence of rational functions commonly known as the Padé approximates or the Gaussian convergents. These approximations are related to the asymptotic expansion for $-Ei(-z)$, see (1.7), and for the same number of terms give much more accuracy than the asymptotic expansion. Indeed, the rational approximates can be used effectively in regions where the asymptotic expansion is useless. Thus, if the Padé approximates are used to represent $G(p)$, then an approximation to $g(z)$ is readily composed, and we should expect results valid in a much larger domain of z values than that derived using the asymptotic expansions. This is truly the case, and we now take up the procedure using rational approximates.

In [1], we showed that the incomplete gamma function which includes the exponential integral as a special case can be approximated by a sequence of rational approximates together with a remainder term. An asymptotic estimate of the error was developed and used to prove that if the variable z is fixed, $|\arg z| < \pi$, then the sequence of rational approximates converge. That is, the error approaches zero as the order of the approximation n is increased. The asymptotic estimate of the error is quite realistic. However, the error formulation suffered from the fact that it was not uniform in z. This deficiency was corrected in a recent study [10]. We summarize below our findings for the exponential integral. As previously remarked, similar results are available for the incomplete gamma function in the reference cited. We have

$$-ze^{z}Ei(-z) = V_n(z) + T_n(z) \quad ,$$

$$V_n(z) = \frac{E_n(z)}{F_n(z)} \quad , \quad T_n(z) = \frac{U_n(z)}{F_n(z)} \quad , \qquad (3.1)$$

where

$$E_n(z) = \sum_{k=o}^{n} \frac{(-n)_k}{(k+1)!}\, {}_2F_2\left(\begin{matrix}-n+k,1\\ 1+k,2+k\end{matrix}\middle| -z\right)$$

$$= \sum_{k=o}^{n} (-)^k z^k \sum_{r=k}^{n} \frac{(-)^r \binom{n}{r} (r-k)!}{(r+1)!} \quad , \tag{3.2}$$

and $F_n(z)$ is the ${}_2F_2$ in (3.2) with $k=o$ whence

$$F_n(z) = {}_1F_1(-n;2;-z) \tag{3.3}$$

which when suitably normalized is the Laguerre polynomial $L_n^{(1)}(-z)$. Also

$$U_n(z) = -ze^z \int_z^{\infty} (t-z)^n t^{-n-2} e^{-t} dt = -n!\psi(n+1,0;z) \tag{3.4}$$

where $\psi(a,c;z)$ is a confluent hypergeometric function, and

$$U_n(z) \sim -n!z^{-n-1}{}_2F_0(n+1,n+2;-z^{-1}) \quad ; \quad |z|\to\infty \quad , \quad |\arg z| < 3\pi/2 \quad . \tag{3.5}$$

Both $E_n(z)$ and $F_n(z)$ satisfy the recurrence equation

$$(n+2)F_{n+1}(z) = (z+2n+2)F_n(z) - F_{n-1}(z) \quad . \tag{3.6}$$

Some further useful relations are

$$\left[zD^2+(z+2)D-n\right] F_n(z) = 0 \quad , \quad D = d/dz \quad , \tag{3.7}$$

$$zDF_n(z) = n\left[F_n(z)-F_{n-1}(z)\right] \quad , \tag{3.8}$$

$$zDE_n(z) = (z+n+1)E_n(z) - nE_{n-1}(z) - zF_n(z) \quad . \tag{3.9}$$

Let

$$z = 4(n+1)\sinh^2 \alpha \tag{3.10}$$

where α is real and positive if z is real and positive. To be explicit, it is convenient to write

$$\begin{aligned} &\sinh^2 \alpha = \rho e^{i\theta} \quad , \quad \rho > 0 \quad , \quad \left|\theta\right| < \pi \quad , \\ &\alpha = \beta+i\delta \quad , \quad \beta \text{ and } \delta \text{ real} \quad , \quad \beta > 0 \quad , \quad \left|\delta\right| < \pi/2 \quad , \end{aligned} \tag{3.11}$$

so that

$$\cosh \beta = \left[\frac{1+\rho+ \left\{(1+\rho)^2-4\rho \sin^2 \theta/2\right\}^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}} \quad ,$$

$$\sin \delta \cosh \beta = \rho^{\frac{1}{2}} \sin \theta/2 \quad , \quad \cos \delta \sinh \beta = \rho^{\frac{1}{2}} \cos \theta/2 \quad . \tag{3.12}$$

Then

$$T_n(z) \sim \frac{-2\pi z e^{z-2(n+1)(2\alpha+\sinh 2\alpha)}\left[1+ \frac{P_1(\alpha)}{n+1} + \frac{P_2(\alpha)}{(n+1)^2} +O(n^{-3})\right]}{\left[1- \frac{P_1(\alpha)}{n+1} + \frac{P_2(\alpha)}{(n+1)^2} +O(n^{-3})\right]} \quad ,$$

$$P_1(\alpha) = (96 \coth^3 \alpha)^{-1}\left[9 \coth^4\alpha -6 \coth^2\alpha +5\right] \quad , \tag{3.13}$$

Equation (3.13) concluded on next page.

$$P_2(\alpha) = (18432 \coth^6 \alpha)^{-1}\left[-135 \coth^8\alpha + 180 \coth^6\alpha + 558 \coth^4\alpha - 924 \coth^2\alpha + 385\right], \tag{3.13}$$

uniformly in z for n large and $|\arg z| < \pi$. Under these conditions it is clear that

$$\lim_{n\to\infty} T_n(z) = 0 \quad . \tag{3.14}$$

Thus the rational approximation converges everywhere in the complex plane except when z is on the negative real axis. This is to be expected since the zeros of the Laguerre polynomial $F_n(z)$ lie on the negative real axis, and obviously the rational approximates cannot be used within sufficiently small neighborhoods of these zeros. As a remark aside, in [10] it is shown that, if n is fixed, $z = -x$, and x is sufficiently large, then $V_n(-x)$ may be used to approximate $xe^{-x}Ei(x)$. Thus the asymptotic expansion for $xe^{-x}Ei(x)$, see (1.7), and $V_n(-x)$ exhibit the same behavior. Indeed, with n fixed, we can show that

$$\lim_{x\to\infty} x^{2n}T_n(-x) = 0 \quad .$$

This follows from an asymptotic representation for $T_n(-x)$ which is similar in form to (3.13). For further details, see [10].

Another rational approximation for the exponential integral is given by

$$-ze^zEi(-z) = S_n(z)+R_n(z) \quad ,$$

$$S_n(z) = \frac{M_n(z)}{N_n(z)} \quad , \quad R_n(z) = \frac{W_n(z)}{N_n(z)} \quad , \tag{3.15}$$

where

$$M_n(z) = -z \sum_{k=0}^{n-1} \frac{(-n)_{k+1}}{(k+1)!(k+1)} \, {}_2F_2\left(\begin{matrix} -n+k+1, \; 1 \\ 2+k, 2+k \end{matrix} \middle| -z \right)$$

$$= \sum_{k=1}^{n} (-)^k z^k \sum_{r=k}^{n} \frac{(-)^r \binom{n}{r} (r-k)!}{r!} \quad , \tag{3.16}$$

and $N_n(z)$ is the ${}_2F_2$ in (3.16) with $k = -1$, so that

$$N_n(z) = {}_1F_1(-n;1;-z) \quad . \tag{3.17}$$

Also

$$W_n(z) = ze^z \int_z^{\infty} (t-z)^n t^{-n-1} e^{-t} dt = n! z \psi(n+1,1;z) \quad . \tag{3.18}$$

Both $M_n(z)$ and $N_n(z)$ satisfy the recurrence equation

$$(n+1)N_{n+1}(z) = (z+2n+1)N_n(z) - nN_{n-1}(z) \quad , \tag{3.19}$$

and (3.9) is valid if $E_n(z)$ and $F_n(z)$ are replaced by $M_n(z)$ and $N_n(z)$, respectively. Also, $N_n(z)$ satisfies (3.8), and

$$\left[zD^2 + (z+1)D - n \right] N_n(z) = 0 \quad . \tag{3.20}$$

A formula for $R_n(z)$ which resembles (3.13) can be given. However, it is convenient to exhibit a formula which depicts the relation between $R_n(z)$ and $T_n(z)$. Thus

$$R_n(z) \sim -T_n(z)\exp\left[2\alpha + \frac{\tanh\alpha}{4(n+1)} + \frac{(\tanh\alpha)(2\cosh^2\alpha+1)}{48(n+1)^2\cosh^2\alpha} + O(n^{-3})\right]$$

$$\times\left[1 - \frac{\coth\alpha}{4(n+1)} + \frac{\coth^2\alpha}{32(n+1)^2} + \frac{1}{(n+1)^2}\left\{P_1(\alpha) - \frac{\coth\alpha}{8}\right.\right.$$

$$\left.\left. + \frac{1}{64\cosh^2\alpha}\left[\coth\alpha - 2\tanh\alpha + 5\tanh^3\alpha\right]\right\} + O(n^{-3})\right] , \qquad (3.21)$$

whence $R_n(z)$ enjoys the same convergence properties as $T_n(z)$.

With these preliminaries out of the way, we now pass on to application of the rational approximates of $Ei(-z)$ to approximate the solution of (1.1). Combining (1.2) and (3.1), we can write

$$G(p) = G_n(p) - \frac{\frac{1}{2}T_n(2p)}{\left[2+2pe^{2p}Ei(-2p)\right]\left[2-V_n(2p)\right]} ,$$

$$G_n(p) = \frac{1}{2}\frac{\left[F_n(2p)-E_n(2p)\right]}{\left[2F_n(2p)-E_n(2p)\right]} . \qquad (3.22)$$

Thus $G_n(p)$ is a rational approximation to $G(p)$. Let $g_n(z)$ be the inverse transform of $G_n(p)$. Then $g_n(z)$ is an approximation to $G(z)$, and

$$G(z) = g_n(z) + \epsilon_n(z) ,$$

$$\epsilon_n(z) = -(4\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{e^{zp}T_n(2p)dp}{\left[2+2pe^{2p}Ei(-2p)\right]\left[2+2pe^{2p}Ei(-2p)+T_n(2p)\right]} . \qquad (3.23)$$

The numerator and denominator polynomials of $G_n(p)$ follow from (3.2). Clearly they are easy to generate since each satisfies (3.6) with z replaced by $2p$.

$G_n(p)$ is readily decomposed into partial fractions, and by inversion $g_n(z)$ is a sum of exponential functions. Thus

$$G_n(p) = \sum_{k=1}^{n} \frac{a_k}{p+\lambda_k} \quad ,$$

$$g_n(z) = \sum_{k=1}^{n} a_k e^{-\lambda_k z} \quad , \quad g_n(0) = 1/4 \quad . \tag{3.24}$$

Another approximation follows from (3.15), and for this, the equations (3.22) - (3.24) hold with obvious change of notation.

We now prove that the approximation process is convergent. That is, if z is fixed, $|\arg z| < \pi$, then $\lim\limits_{n\to\infty} \epsilon_n(z) = 0$. We have already shown that $h(p)$, see (1.9), has no zeros in the complex plane cut along the negative real axis, and obviously there are no zeros in the right half plane. Let c be fixed. If c is sufficiently large, the asymptotic expansion of $h(p)$ which follows from (1.6) - (1.7) and the asymptotic form of $T_n(z)$, see (3.13), show that $2h(p) + T_n(2p)$ has no zeros on or outside a semicircle of radius c with center at the origin which lies in the right half plane. Indeed, since

$$2h(p) \sim 1 + \frac{1}{2p} - \frac{1}{2p^2} + O(p^{-3}) \quad , \quad |p|\to\infty \quad , \quad |\arg p| < 3\pi/2 \quad ,$$

to prove convergence, it is sufficient to consider the integral

$$\delta_n(z) = - i \lim_{R\to\infty} \int_{c-ik}^{c+ik} \Delta_n(p)dp \quad ,$$

$$\Delta_n(p) = pe^{p(z+2)-2(n+1)(2\alpha+\sinh 2\alpha)} \quad , \quad \sinh^2 \alpha = \frac{p}{2(n+1)} \quad . \tag{3.25}$$

Let C_1 be the path of a quarter circle with center at the origin and radius $c_1 \geq c$ which lies in the upper right half plane. Let C_2 be the imaginary axis extending from the point ic_1 to iR . Then

$$\delta_n(z) = I_1 + I_2 \quad ,$$

$$I_1 = 2\mathrm{Re}\left\{ -i \int_{C_1} \Delta_n(p)dp \right\} \quad , \quad I_2 = 2\mathrm{Im}\left\{ \lim_{R\to\infty} \int_{C_2} \Delta_n(p)dp \right\} \tag{3.26}$$

where Re stands for the real part and Im stands for the imaginary part.

We consider I_1 first. Along C_1 , $p = c_1 e^{i\theta}$ so that

$$I_1 = 2c_1^2 \,\mathrm{Re} \int_0^{\pi/2} \exp\left\{ 2i\theta + c_1(z+2)e^{i\theta} - 2(n+1)(2\alpha + \sinh 2\alpha) \right\} d\theta \quad ,$$

$$\sinh^2 \alpha = \frac{c_1 e^{i\theta}}{2(n+1)} \quad , \quad 0 \leq \theta \leq \pi/2 \quad . \tag{3.27}$$

We now let n be fixed but large with respect to c_1 . Then

$$\cosh \alpha = \left[1 + \frac{c_1 e^{i\theta}}{2(n+1)} \right]^{\frac{1}{2}} = 1 + \frac{c_1 e^{i\theta}}{4(n+1)} - \frac{c_1^2 e^{2i\theta}}{32(n+1)^2} + \ldots \quad ,$$

$$\sinh 2\alpha = 2\left[\frac{c_1 e^{i\theta}}{2(n+1)} \right]^{\frac{1}{2}} \left[1 + \frac{c_1 e^{i\theta}}{4(n+1)} + O(n^{-2}) \right] \quad ,$$

$$2\alpha = 2\left[\frac{c_1 e^{i\theta}}{2(n+1)} \right]^{\frac{1}{2}} \left[1 - \frac{c_1 e^{i\theta}}{12(n+1)} + O(n^{-2}) \right] \quad , \tag{3.28}$$

and

$$2(n+1)(2\alpha+\sinh 2\alpha) = 4\left[2c_1(n+1)\right]^{\frac{1}{2}} e^{i\theta/2}\left[1 + \frac{c_1 e^{i\theta}}{12(n+1)} + O(n^{-2})\right] \tag{3.29}$$

Since $0 \le \rho \le \pi/2$, we have

$$\left|\mathrm{Re}\left\{2(n+1)(2\alpha+\sinh 2\alpha)\right\}\right| \le 4\left[c_1(n+1)\right]^{\frac{1}{2}}\left[1 + O(n^{-1})\right] \quad , \tag{3.30}$$

and as z and c_1 are fixed, it is clear that $\lim\limits_{n\to\infty} I_1 = 0$.

For I_2 , we let $p = iq$ along C_2 . Then

$$I_2 = -2\mathrm{Im}\left\{\lim_{R\to\infty}\int_{c_1}^{R} qe^{iq(z+2)-2(n+1)(2\alpha+\sinh 2\alpha)}dq\right\} \quad ,$$

$$\sinh^2 \alpha = \frac{qe^{i\pi/2}}{2(n+1)} \quad . \tag{3.31}$$

We suppose that $0 \le \arg z < \pi$ so that in (3.31), $e^{iq(z+2)}$ is always bounded for $c_1 \le q \le \infty$. It is convenient to split the range of integration into two parts. The first from c_1 to $R_1 = 4(n+1)$ and the second from R_1 to R . For convergence, it is therefore sufficient to consider the integrals

$$I_3 = \mathrm{Re}\left\{\int_{c_1}^{R_1} qe^{-2(n+1)(2\alpha+\sinh 2\alpha)}dq\right\} \quad ,$$

$$I_4 = \mathrm{Re}\left\{\int_{R1}^{R} qe^{-2(n+1)(2\alpha+\sinh 2\alpha)}dq\right\} \quad . \tag{3.32}$$

Now apply (3.11) - (3.13) with $\theta = \pi/2$ and $\rho = q/2(n+1)$. Then

$$2(n+1)\mathrm{Re}(\sinh 2\alpha) = 2\left[(n+1)q\right]^{\frac{1}{2}} \left[(1+\rho^2)^{\frac{1}{2}}-\rho\right] , \tag{3.33}$$

$$e^{-2(n+1)\mathrm{Re}(2\alpha)} = \left[(1+\rho^2)^{\frac{1}{2}}+\rho+2^{\frac{1}{2}}\rho^{\frac{1}{2}}\left\{(1+\rho^2)^{\frac{1}{2}}+\rho\right\}^{\frac{1}{2}}\right]^{-2n-2} \tag{3.34}$$

For the evaluation of I_3 , $0 < \dfrac{c_1}{2(n+1)} \le \rho \le 2$. In this case

$$e^{-2(n+1)\mathrm{Re}(2\alpha)} < \left\{1+\left[\frac{c_1}{2(n+1)}\right]^{\frac{1}{2}}\right\}^{-8(n+1)}$$

$$= \left\{1+\left[\frac{c_1}{2(n+1)}\right]^{\frac{1}{2}}\right\}^{\left[\frac{2(n+1)}{c_1}\right]^{\frac{1}{2}}(-4)\left[2c_1(n+1)\right]^{\frac{1}{2}}} \tag{3.35}$$

Now $\lim\limits_{x\to\infty} (1+x^{-1})^{ax} = e^a$, and so

$$\lim_{n\to\infty} I_3 < \frac{1}{2}\lim_{n\to\infty}\left[16(n+1)^2-c_1^2\right]e^{-4\left[2c_1(n+1)\right]^{\frac{1}{2}}} = 0 \quad . \tag{3.36}$$

In I_4 , let $R\to\infty$ and put $q = 2(n+1)t$. Then

$$I_4 = 4(n+1)^2 \mathrm{Re} \left\{ \int_2^\infty t e^{-2(n+1)(2\alpha+\sinh 2\alpha)} dt \right\} , \quad \sinh^2 \alpha = t e^{i\pi/2} . \tag{3.37}$$

Thus (3.33) and (3.34) apply with $\rho = t$ and $2 \le t \le \infty$. In this case

$$e^{-2(n+1)\mathrm{Re}(2\alpha)} < (4t)^{-2n-2} . \tag{3.38}$$

Also $2(n+1)\mathrm{Re}(\sinh 2\alpha) \ge 0$ for all t , $2 \le t \le \infty$. Hence

$$I_4 < \frac{(n+1)^2}{2^{2n+2}} \int_2^\infty \frac{dt}{t^{2n+1}} = \frac{(n+1)^2}{n\cdot 2^{4n+3}} ,$$

$$\lim_{n\to\infty} I_4 = 0 , \tag{3.39}$$

and the proof of convergence is complete. A similar argument shows that the approximation technique based on (3.15) also converges. In this connection, see (3.21).

We have attempted to develop a useful and realistic approximation for $\epsilon_n(z)$ so that one could determine, a priori, for a given z the value of n required to achieve a given level of accuracy. Some preliminary results in this direction are available, but the analysis is far from complete, and we defer further comments for a later report. It is pertinent to remark, however, that numerical values determined by the approximation technique can be used to give pragmatic estimates of the error. This and much more are brought out in the following section.

In Table I, we tabulate $\int_z^\infty t^{-1}e^{-t}dt$, its rational approximation (see (3.1)), the exact error, and the approximate error according to (3.13). The data are developed for $z = 2e^{i\varphi}$ and $n = 4$. Note that the error formula for $T_n(z)$ is quite remarkable for it is very realistic even for small n .

TABLE I

$z = 2e^{i\varphi}$ φ	$\int_z^\infty t^{-1}e^{-t}dt$	$z^{-1}e^{-z}V_4(z)$
0	$4.89005\cdot 10^{-2}$	$4.89190\cdot 10^{-2}$
$\pi/4$	$-3.95846\cdot 10^{-2}-8.22921i\cdot 10^{-2}$	$-3.95652\cdot 10^{-2}-8.22414i\cdot 10^{-2}$
$\pi/2$	$-4.22981\cdot 10^{-1}+3.461671i\cdot 10^{-2}$	$-4.23980\cdot 10^{-1}+3.43682i\cdot 10^{-2}$
$3\pi/4$	$-2.16947 \quad +0.31777i$	$-2.12827 \quad +0.36462i$

$z = ze^{i\varphi}$ φ	$z^{-1}e^{-z}T_4(z)$ Exact	$z^{-1}e^{-z}T_4(z)$ Approximate
0	$-0.185\cdot 10^{-4}$	$-0.184\cdot 10^{-4}$
$\pi/4$	$-0.194\cdot 10^{-4}-0.507i\cdot 10^{-4}$	$-0.194\cdot 10^{-4}-0.507i\cdot 10^{-4}$
$\pi/2$	$0.999\cdot 10^{-3}+0.248i\cdot 10^{-3}$	$0.996\cdot 10^{-3}+0.249i\cdot 10^{-3}$
$3\pi/4$	$-0.412\cdot 10^{-1}-0.468i\cdot 10^{-1}$	$-0.412\cdot 10^{-1}-0.473i\cdot 10^{-1}$

In Table II, we record the values of a_k and λ_k for $n = 3(1)6$, and in Table III we compare $g_n(z)$ with the "exact" $g(z)$ for several values of z. These data are based on the representation (3.1). Tables IV and V are the same as I and II, respectively, save that the approximations are based on (3.15). Here we use the notation a_k^*, λ_k^* and g_n^* , which parallels that of (3.24). The "exact" values for $z \le 5$ are taken from a study by Fox and Goodwin [11], who use (1.1) in a study of numerical solutions to integral equations. The value of $g(z)$ for $z = 6$ was found from (1.13) by numerical integration. It appears that the value of $g(z)$ for $z = 5$ is a unit too large in the last place. In Tables III and V, we also give values of $\delta(z)$ for $z = 4(1)6$ which are derived from the $n = 4,5$ and 6 approximates using Aitken's δ^2-process.

TABLE II

n = 3

k	a_k	λ_k
1	0.09609 77	-0.71098 97
2	0.14655 50	-1.89746 62
3	0.00734 73	-3.89154 41

n = 4

k	a_k	λ_k
1	0.05937 40	-0.53891 09
2	0.15682 37	-1.56937 11
3	0.03138 23	-2.91402 74
4	0.00042 00	-5.47769 05

n = 5

k	a_k	λ_k
1	0.03979 98	-0.42941 92
2	0.14295 32	-1.33416 55
3	0.06412 69	-2.40159 29
4	0.00309 96	-4.20473 33
5	0.00002 05	-7.13008 91

n = 6

k	a_k	λ_k
1	0.02843 62	-0.35499 71
2	0.11865 90	-1.15071 93
3	0.09292 31	-2.07864 32
4	0.00973 90	-3.47492 24
5	0.00024 17	-5.61773 46
6	0.00000 10	-8.82298 34

TABLE III

z	$g_3(z)$	$g_4(z)$	$g_5(z)$	$g_6(z)$	$\delta(z)$	$g(z)$
0	0.25000	0.25000	0.25000	0.25000		0.25000
0.5	0.12515	0.12515	0.12515	0.12515		0.12515
1.0	0.06932	0.06941	0.06941	0.06941		0.06942
1.5	0.04161	0.04193	0.04197	0.04197		0.04197
2.0	0.02648	0.02718	0.02730	0.02732		0.02733
3.0	0.01188	0.01322	0.01363	0.01374	0.01378	0.01378
4.0	0.00567	0.00718	0.00783	0.00809	0.00826	0.00821
5.0	0.00267	0.00407	0.00483	0.00520	0.00555	0.00547
6.0	0.00135	0.00235	0.00307	0.00350	0.00414	0.00391

TABLE IV

n = 3		
k	a_k^*	λ_k^*
1	0.03668 99	-0.31385 93
2	0.18750 00	-1.50000 00
3	0.02581 01	-3.18614 07

n = 4		
k	a_k^*	λ_k^*
1	0.02025 70	-0.22504 61
2	0.15173 03	-1.18873 16
3	0.07630 34	-2.38574 42
4	0.00170 92	-4.70047 81

n = 5		
k	a_k^*	λ_k^*
1	0.01279 44	-0.17429 89
2	0.11138 06	-0.96597 23
3	0.11584 45	-1.98033 70
4	0.00989 00	-3.55882 82
5	0.00009 05	-6.32056 36

n = 6		
k	a_k^*	λ_k^*
1	0.00880 86	-0.14178 99
2	0.08145 38	-0.80353 59
3	0.13290 86	-1.71581 14
4	0.02597 15	-2.92720 02
5	0.00085 32	-4.92022 14
6	0.00000 43	-7.99144 12

TABLE V

z	$g_3^*(z)$	$g_4^*(z)$	$g_5^*(z)$	$g_6^*(z)$	$\delta^*(z)$	$g(z)$
0	0.25000	0.25000	0.25000	0.25000		0.25000
0.5	0.12518	0.12515	0.12515	0.12515		0.12515
1.0	0.06971	0.06943	0.06941	0.06941		0.06942
1.5	0.04289	0.04210	0.04197	0.04197		0.04197
2.0	0.02896	0.02764	0.02738	0.02733		0.02733
3.0	0.01639	0.01466	0.01403	0.01384	0.01376	0.01378
4.0	0.01092	0.00955	0.00875	0.00841	0.00813	0.00821
5.0	0.00774	0.00697	0.00625	0.00582	0.00524	0.00547
6.0	0.00560	0.00537	0.00484	0.00442	0.00282	0.00391

As previously remarked, our formulation of the error in the approximation process is not in a convenient form for practical use. Since the approximate solutions converge and are easy to generate, it is suggested that the numbers themselves be employed to apprise the accuracy of the process. Thus if two or more successive convergents are evaluated, one can accept the common digits as correct. Further, as shown in the above tables, Aitken's δ^2-process is valuable provided n, the order of the approximation, is sufficiently large. Note that in Tables II and IV, the δ^2-process gives improved values except for $z = 6$ in Table IV.

Another technique useful to assess the error is to compare the approximate entries for large z with values deduced from the asymptotic expansion. If we use all the terms given in (2.6), then for $z = 5$ and 6, we get 0.00509 and 0.00374, respectively. In our present example, the asymptotic representation is deficient since an error estimate is not available, and it is clear from the calculations that the asymptotic expansion is not sufficiently reliable unless z is considerably larger than 6. Of course, had we computed approximates for larger n, then we could take z larger whence values deduced from the asymptotic representation would be more informative.

In our study, a further procedure to get improved values is to average the corresponding entries in Tables III and V. This works because in Table III the values are converging from below while in Table V, the values are converging from above. Thus for $z = 5$ and 6, we get the improved values 0.00551 and 0.00396, respectively. Of course, in practice it may not always be possible to have rational approximations so that one can get convergence from both sides. In any event, it seems that using only one sequence of rational approximates with more entries for n would be more informative.

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